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CRITICAL AND NEAR-CRITICAL VALUES IN POLYNOMIAL CONTROL PROBLEMS, I: ONE-DIMENSIONAL CASE

BY

M. BRISKIN

The Jerusalem Technological School, Jerusalem 91076, Israel

AND

Y. YOMDIN

Department of Theoretical Mathematics
The Weizmann Institute of Science, Rehovot 76100, Israel

ABSTRACT

For an input-to-state mapping of a polynomial one-dimensional control system we study the geometry of critical and near-critical controls, trajectories and values. Quantitative bounds, depending only on the degree of the polynomial involved, are obtained. Examples are considered, showing these bounds to be essentially sharp.

1. Introduction

In this paper we start a detailed presentation of the results, announced in [1], [3].

Consider a control problem of the form

$$\dot{x} = f(x, u), \qquad x(0) = x^0,$$

(1) where $x \in \mathbb{R}^n$, $u(t) \in U \subseteq \mathbb{R}^m$, with U a compact subset of $\mathbb{R}^m$.

Let $T > 0$ be fixed. The input-to-state mapping $\mathcal{T} : u \to x_u(T)$ of (1) associates to each control $u : [0,T] \to U$ the state $\mathcal{T}(u) = x_u(T)$, to which u steers

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the system from the initial state x^0 in time T. ($x_u(t)$ denotes the corresponding solution of (1).)

The study of the geometry of this mapping and, in particular, the geometry of the closure Ω_T of its image (the time T-reachable set of (1)) is one of the central problems in control theory (see, e.g., [13–17]). Of particular importance are the boundary points of Ω_T, and the corresponding extremal trajectories.

It is well-known that the behavior of extremal trajectories can be very complicated. They may have an infinite number of control jumps in a generic way ([8]), the time reachable set Ω_T may have an infinite number of "holes" ([10]), etc.

In [1], [3] we started the study of a special class of trajectories, which we call extremal (near-extremal, respectively) of rank zero. Geometrically, these trajectories lead to the "vertices" (the points of a high curvature, respectively) of the boundary $\partial\Omega_T$.

Alternatively, the (precisely) extremal rank zero trajectories can be characterized as follows: for such a trajectory $x_u(t)$ there is not direction in $\mathbb{R}^n$ in which we can move the endpoint $x_u(T)$ by infinitesimal control variations both in this direction and in the opposite one.

The rank zero trajectories have some important properties, which are not shared by general extremal trajectories:

(i) The translation of the rank zero property into the local behavior of $x_u(t)$ (in the spirit of the maximum principle) produces strong necessary conditions, not involving dual variables. Consequently, rank zero trajectories follow certain "algebraic" patterns in the phase space.

(ii) In many cases the phase space can be, in fact, subdivided into a finite number of parts, and any rank zero trajectory can visit each of these parts at most a finite (and effectively bounded) number of times.

In particular, in these cases only a finite number of "essential" control jumps are possible.

The conclusion for a time T reachable set is that the set of its vertices is finite.

Near-extremal trajectories appear quite naturally as a result of a "finite accuracy relation" of extremality conditions: an extremal trajectory is characterized by impossibility to move the endpoint infinitesimally in certain directions. For a corresponding near-extremal trajectory the infinitesimal control action in the forbidden directions must be small (but, possibly, nonzero). Below we give precise definitions for near-critical (instead of near-extremal) trajectories.

The study of near-critical trajectories is important from theoretical and applied points of view: only near-criticality can be effectively imposed or verified in computations with finite accuracy.

The behavior of near-critical trajectories is much more complicated than of critical ones, since the control may cover subsets with the nonempty interior in the phase space. However, we show that the algebraicity and finiteness of rank zero critical trajectories are preserved for near-critical ones "topologically":

(a) Two rank zero near-critical trajectories which are of the same topological type (in a proper sense) are in fact close to one another.
(b) A topological type of trajectory is determined by a finite number of choices of the control behavior.

As an ultimate result we obtain that the set of the endpoints of rank 0 near-extremal trajectories (in particular, the set of high curvature points on $\partial\Omega_T$) is small.

In [1] and [3] only statements of the results and some sketches of the proof are given. In the present paper we start the detailed presentation and the proof of these results, in the simplest situation: we consider one-dimensional control systems and allow only Lipschitzian controls. However, the analysis and the geometry of this simplest case reflect the difficulties of a general situation, and the techniques, developed in this paper, form an important ingredient in [4], [5], where we treat higher-dimensional cases and discontinuous controls.

Also, the notion of an extremal trajectory we replace here by a closely related, but somewhat simpler notion of a critical one.

Critical controls are those $u(t)$ for which the differential $d\mathcal{T}_u$ of the input-to-state mapping $\mathcal{T}$ of (1) vanishes. (We assume controls to belong to the space $W = L^p[0,T], p > 1$. Under this assumption $\mathcal{T}$ is Fréchet differentiable at any $u \in W$; see, e.g., [14], [6].) A corresponding trajectory $x_u(t)$ is called a critical trajectory and $x_u(T) = \mathcal{T}(u) \in \mathbb{R}^n$ is called a critical value of $\mathcal{T}$.

Near-critical controls, trajectories and values are parametrized by $\gamma \geq 0 : u$ is called γ-critical if $\|d\mathcal{T}_u\| \leq \gamma$. $x_u(t)$ and $x_u(T) = \mathcal{T}(u)$ are called a γ-critical trajectory and a γ-critical value, respectively.

Critical trajectories are those, along which the linearized system is not completely controllable. They are intensively studied (see [13], [14]). For example, the control strategy described in [13] applies first a relatively simple "open loop" control generator to steer the system approximately into the desired state. To regulate for small derivations one uses then linear control design techniques for a linearized system along the initial trajectory. But to regulate successively one needs this trajectory to be not only nonsingular (completely controllable) but to be "far away" enough from singular ones: in numerical computations very small jacobians and zero jacobians are indistinguishable. The situation is quite similar in any application where finite accuracy computations are involved (see, e.g., [19]).

Our results provide exactly this type of "quantitative" information. They show that under certain conditions the set of near-critical values is small. Hence, for most of the states, any trajectory leading to this state is well separated from singular ones.

Let $W_k \subseteq W = L^p[0,1]$ be the set of K-Lipschitzian controls $u(t)$ with $|u(t)| \leq 1, t \in [0,1]$ (we put $T = 1$). Denote by $\Delta(\mathcal{T},\gamma)$ the set of all γ-critical values of $\mathcal{T}$ on W_K. Let $n = m = 1$ and let $f(x,u)$ in (1) be a polynomial of degree d, $|f(x,u)| \leq 1$ for $|u| \leq 1, |x| \leq 1$.

THEOREM 1.1: *For any $K > 0, \gamma \geq 0, \nabla(\mathcal{T},\gamma)$ can be covered by $C_1(d)$ intervals of length $C_2(d,K)\gamma^{q/(q+1)}$ $(1/p + 1/q = 1)$, where*

$$C_1(d) = 8^3 d^6 2^{2(d+1)^2},$$
$$C_2(d,K) = 4e^{d^2}[(q+1)d^4(1+K)e^{2qd^2}]^{1/(q+1)}.$$

In particular, the number of critical ($\gamma = 0$) values of $\mathcal{T}$ on $\bigcup_{K>0} W_K$ does not exceed $C_1(d)$.

In section 4 below we give examples showing these estimates to be essentially sharp.

The result of Theorem 1.1 concerns the question of validity of Sard's theorem and its quantitative version ([12], [19]) for a mapping $\mathcal{T}$. This question is important since Sard's theorem and its extensions provide a powerful tool in nonlinear analysis. On the other hand, the extent to which Sard's theorem-like results are satisfied for a certain mapping measures the complexity of this mapping (see [18]). While generally fine in smooth finite-dimensional situation, these results are usually violated for infinite-dimensional spaces, even for C^∞-mappings ([7], [18]). The known exclusions are rare: Fredholm mappings and variational problems with "finite index", for which the problem is, essentially, reduced to a finite-dimensional one, seem to be the only known general examples. Some new examples are given in [18].

The input-to-state mapping $\mathcal{T}$ of (1) does not belong to the above classes. However, Theorem 1.1 shows that for $n = 1$ and f a polynomial, Sard's theorem and its quantitative extension are true.

An overexponential growth of $C_1(d)$ in d reflects a high complexity of the mapping $\mathcal{T}$. According to the "approximative complexity" approach of [18] it indicates a possibility to find "almost analytic" one-dimensional control problems, for which Sard's theorem is violated. Such examples are indeed constructed in [2].

In section 2 we state the main result in a slightly generalized form, discuss a simple example and outline the techniques used. In section 3 proofs are given. Finally, in section 4 we show our bounds to be sharp, considering corresponding examples.

2. Statement of the Main Results, Examples, and Outline of Proofs

We consider non-linear one-dimensional control systems

$$\dot{x}(t) = f(x(t), u(t)), \qquad x(0) = x_0,$$

where $x(t) \in R$ and $u(t) \in R$ is the control, at time $t \in [0,1]$.

Let W be a space of admissible controls, consisting of Lipschitz functions on [0, 1]:

$$W = \bigcup_{K>0} W_K,$$

where for $K > 0$

$$W_K = \{u(\cdot) \,\big|\, |u(t_2) - u(t_1)| \le K|t_2 - t_1|, t_1, t_2 \in [0,1]\}.$$

We define a functional $J\colon W \in R$ by

$$J(u(\cdot)) = x(1) \tag{1}$$

where $x(\cdot)$ is a solution of a differential equation

$$\dot{x} = f(x, u(t)), \qquad x(0) = x_0, \quad t \in [0,1]. \tag{2}$$

Below we make assumptions, sufficient to guarantee the existence of solutions (2) for $t \in [0,1]$.

A pair $(x(\cdot), u(\cdot))$ is called an admissible trajectory. Geometrically, an admissible trajectory is a parametric curve in the (x, u) plane.

Definition 1: A control $u(\cdot)$ is called critical, if

$$DJ_u \equiv 0, \text{ where } DJ_u(v) = \lim_{\alpha \to 0} \frac{1}{\alpha}[J(u + \alpha v) - J(u)].$$

(Compare with the conditions for the Fréchet differentiability of J, given in [14].)

Let $x(t, \alpha)$ be a solution of the equation

$$\dot{x} = f(x, u(t) + \alpha v(t)), \qquad x(0) = x_0,$$

where α is a parameter. Then

$$DJ_u(v) = \frac{\partial x}{\partial \alpha}(1, 0).$$

But the derivative with respect to the parameter $z(t) = \partial x / \partial \alpha$ is the solution of a linearized equation

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}(x(t), u(t))z + \frac{\partial f}{\partial u}(x(t), u(t))v(t), \qquad z(0) = 0. \tag{3}$$

Then

$$z(1) = DJ_u(v) = F(1) \int_0^1 F^{-1}(\tau) \frac{\partial f}{\partial u}(x(\tau), u(\tau))v(\tau)d\tau, \tag{4}$$

where

$$F(t) = \exp \int_0^t \frac{\partial f}{\partial x} d\tau$$

is the solution of a homogeneous equation

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}(x(t), u(t))z.$$

From this we immediately obtain the following:

COROLLARY 1: *A control $u(\cdot)$ is critical if and only if*

$$\frac{\partial f}{\partial u}(x(t)) \equiv 0, \qquad t \in [0, 1].$$

The space of admissible controls W can be considered with various norms. In this paper we consider only the L_p-norms on W, but the following definition makes sense for any fixed norm $\| \quad \|$ on W:

Definition 2: A control $u(\cdot)$ is called γ-critical, if $\|DJ_u\| \leq \gamma$. ∎

Now we fix the notation: Let $T_1 > 0$ and $T_2 > 0$ be fixed.

(1) $\tilde{\Sigma}(J,\gamma) \subseteq W$ is a set of γ-critical controls for which the curve $(x(\cdot), u(\cdot))$ lies in the rectangle $D = \{|x| \leq T_1, |u| \leq T_2\}$;
$\tilde{\Delta}(J,\gamma) = J(\tilde{\Sigma}(J,\gamma)) \subseteq \mathbf{R}$ is the set of γ-critical values of the functional J on $\tilde{\Sigma}(J,\gamma)$;

(2)
$$M_{ij} = \max_D \left|\frac{\partial^{i+j} f}{\partial^i x \partial^j u}\right|, \qquad 0 \leq i,j \leq 2;$$
$$m_{ij} = \min_D \left|\frac{\partial^{i+j} f}{\partial^i x \partial^j u}\right|, \qquad 0 \leq i,j \leq 2;$$

(3)
$$\Phi_\gamma^{(1)} = \left\{(x,u) \middle| \left|\frac{\partial f}{\partial u}(x,u)\right| \leq \gamma\right\},$$
$$\Phi_\gamma^{(2)} = \left\{(x,u) \middle| \left|\frac{\partial f}{\partial u}(x,u)\right| = \gamma\right\},$$
$$\Phi_\gamma^{(3)} = \left\{(x,u) \middle| \left|\frac{\partial f}{\partial u}(x,u)\right| \geq \gamma\right\},$$
$$\tilde{\Phi}_\gamma^{(i)} = \Phi_\gamma^{(i)} \cap D, \qquad i = 1,2,3.$$

We will always assume $|u(t)| \leq T_2$. The following (not very restrictive) assumption guarantees that any solution of (2) stays in D (and, in particular, exists) for $t \in [0,1]$:
$$|x_0| + M_{00} \leq T_1.$$

Let the norm $\| \quad \|$ on W be chosen as the L_p-norm $\|u\|_p = \left(\int_0^1 |u(t)|^p dt\right)^{1/p}$. We also assume that the controls u belong to W_K, for a fixed $K > 0$.

Now we can state our main result:

THEOREM 1: *Let $f(x,u)$ be a polynomial of degree d. Then for any $\gamma > 0$ the set $\tilde{\Delta}(J,\gamma) \subseteq R$ can be covered by $N(d) = 8^3 d^6 2^2 (d+1)^2$ intervals of length γ', where $\gamma' = C\gamma^{q/q+1}, 1/p + 1/q = 1$, and*
$$C = 4T_2 e^{M_{10}} \cdot [(q+1)(M_{00}M_{11} + M_{02}K)e^{q(m_{10}+M_{10})}]^{1/q+1}.$$

COROLLARY 2 (Sard's theorem): *Under the above assumptions, the number of critical values of J on any W_K does not exceed $N(d)$.*

To prove the corollary it is enough to apply Theorem 1 with $\gamma \to 0$.

The bounds given by Theorem 1 and Corollary 2 are essentially sharp, as the examples in section 4 show.

Remark: The overexponential growth with the degree d of the covering numbers for critical values of polynomial control problems has important consequences for control problems with only smooth f. In particular, examples with f and controls of class C^∞, where Sard's theorem is not true, are abundant. In the following, we plan to present some of these examples, as well as an investigation of the "approximative complexity" of the control problem (2) in the spirit of [18]. ∎

Now consider the following example, which, though simple, illustrates our approach in restricting the class of admissible controls, as well as the techniques used in proofs. Consider the following system:

$$\begin{cases} \dot{x} = u^3 - 3u, \ x(0) = 0, \\ t \in [0,1], \ x, u \in R. \end{cases} \tag{5}$$

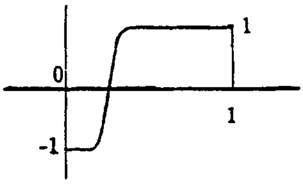
Then

$$J(u) = \int_0^1 [u^3(t) - 3u(t)]dt.$$

A control u is critical for J iff $f_u = 0$ along the corresponding trajectory. Thus any u which assumes only the values -1 and 1 is critical for (5). Now for controls

$$u_\tau = \begin{cases} -1, & 0 \le t \le \tau, \\ 1, & \tau < t \le 1, \end{cases}$$

$\tau \in [0,1], J(u_\tau) = 4\tau - 2$, and hence critical values of J cover the interval $[-2,2]$. (Notice that J is differentiable in any L_p-norm on the control space, $p > 1$ (see [14]), and J is C^∞-smooth in L_∞-norm (see, e.g., [6]). The above family $u_\tau, \tau \in [0,1]$, forms a continuous non-rectifiable curve in any L_p-norm, $1 < p < \infty$. If we require controls u to be continuous, then there are only two critical controls $u \equiv -1$ and $u \equiv 1$, with the corresponding critical values 2 and -2. But any control, switching from -1 to 1 in a sufficiently short time, is almost critical (the L_p-norm of dJ at such controls is of order $\delta^{1/q}$, where δ is the switching interval and $1/p + 1/q = 1$). Hence, for any $\gamma > 0$, the γ-critical values of J on the space of continuous controls cover the interval $[-2, 2]$: all the controls u of the form



are γ-critical, if the switching is fast enough.

Now assume that the slopes of admissible controls are bounded by K. Then the switching time from -1 to 1 cannot be less than $2/K$, and therefore $\|DJ(u)\|_p > (2/K)^{1/q}$, if u switches from the neighborhood of -1 to a neighborhood of 1. Hence for $\gamma \leq (2/K)^{1/q}$, any γ-critical control must stay either near -1 or near 1. Respectively, γ-critical values will form two intervals around -2 and 2, of size $\sim \gamma^{1/q+1}$.

This example illustrates one of the basic geometric observations in our approach: assuming controls to be K-Lipschitzian, we can guarantee any γ-critical trajectory to lie in a set $|f_u| \leq \gamma_1$, where γ_1 depends on K and γ, and tends to 0 as $\gamma \to 0$ (see Proposition 1 below for detailed computations. In example (5) this set consists of two horizontal strips around the lines $u = -1$ and $u = 1$).

Now the second main geometric ingredient in our approach is based on real algebraic geometry. For $f(x, u)$ a polynomial, the set $\Phi^{(1)}_{\gamma_1} = \{(x, u) \| f_u(x, u)| \leq \gamma_1\}$ is defined by polynomial inequalities, i.e. it is semialgebraic. Using metric properties of semialgebraic sets, obtained in [11], we show that any two γ-critical trajectories, which are topologically equivalent as curves in $\Phi^{(1)}_{\gamma_1}$, are, in fact, γ_1-close to one another. (See Definition 1, Lemma 1 and Lemma 2 below. In (5) two trajectories are topologically equivalent if and only if they lie in the same strip.)

This reduces the problem of bounding the geometry of γ-critical values to the topological (combinatorial) problem of classifying topologically different curves in Φ_γ. Here we use real algebraic geometry once more: namely, the bounds for the topological structure of real semialgebraic sets.

Notice also that if we adopt another definition of critical controls, requiring also vanishing of the derivative of J with respect to the jump moment of u, then the controls u_τ above are no longer critical, except for $\tau = 0$ and $\tau = 1$.

3. Proof of Theorem 1

Assume, as above, that the L_p-norm $\| \ \|_p$ is chosen on W, and that all the controls considered belong to W_k for a fixed $K > 0$.

PROPOSITION 1: *If $u(\cdot) \in \tilde{\Sigma}(J,\gamma)$ and $\gamma \le C_1$, then for every $t \in [0,1]$ and $p > 1$ it holds that*

$$\left|\frac{\partial f}{\partial u}(x(t),u(t))\right| \le C_2\gamma^{q/(q+1)},$$

where $1/p + 1/q = 1$ and the constants are

$$C_1 = \left[\frac{e^{-q(m_{10}+M_{10})}}{2q+2}\right]^{1/q} \text{ and } C_2 = [(q+1)(M_{00}M_{11}+M_{02}K)e^{q(m_{10}+M_{10})}]^{1/(q+1)}.$$

Proof: The function

$$\varepsilon(t) = \left|\frac{\partial f}{\partial u}(x(t),u(t))\right|$$

is Lipschitz on [0,1] with a constant $K_1 = M_{00}M_{11} + M_{02}K$. Set

$$M = \max_{[0,1]} \varepsilon(t) = \varepsilon(t_0)$$

and consider the function

$$\varepsilon_1(t) = \begin{cases} M - K_1|t-t_0|, & |t-t_0| \le M/K_1, \\ 0, & |t-t_0| > M/K_1 . \end{cases}$$

It is obvious that $\varepsilon_1(t) \le \varepsilon(t)$ on [0,1]. Then

$$\|DJ_u\|^q = F(1)^q \int_0^1 \left|F^{-1}(\tau)\frac{\partial f}{\partial u}(x(\tau),u(\tau))\right|^q d\tau \ge C\int_0^1 [\varepsilon_1(\tau)]^q d\tau,$$

where $C = e^{-q(m_{10}+M_{10})}$. If $M \ge K_1/2$ then

$$\|KJ_u\|^q \ge C\int_0^{1/2} (K_1\tau)^q d\tau = \frac{C}{2^{(q+1)}(q+1)}K_1^q.$$

Hence for γ sufficiently small, namely

$$\gamma < C_1, \qquad C_1 = \left(\frac{C}{2q+2}\right)^{1/q} \cdot \frac{K_1}{2},$$

it follows from $\|DJ_u\| \leq \gamma$ that $M < K_1/2$. Let now $\gamma < C_1$ and $\|DJ_u\| \leq \gamma$. Then for $t_0 \geq \frac{1}{2}$ we have

$$\gamma^q \geq \|DJ_u\|^q \geq C \int_{t_0 - M/K_1}^{t_0} [M + K_1(\tau - t_0)]^q d\tau = \frac{CM^{q+1}}{K_1(q+1)}.$$

The same bound holds for $t_0 \leq 1/2$ and thus

$$M \leq \left[\frac{K_1(q+1)}{C}\right]^{1/(q+1)} \cdot \gamma^{q/(q+1)} = C_2 \cdot \gamma^{q/(q+1)}. \qquad \blacksquare$$

COROLLARY 2: *For $\gamma < C_1$ and $u(\cdot) \in \tilde{\Sigma}(J, \gamma)$, it holds that*

$$(x(\cdot), u(\cdot)) \in \tilde{\Phi}^{(1)}_{\gamma_1}, \qquad \gamma_1 = C_2 \gamma^{q/(q+1)}.$$

From now on we shall assume that $f(x,u)$ is a polynomial of (total) degree d in x and u. Then the sets $\Phi^{(i)}_\gamma$ are defined by polynomial equations (inequalities) in the (x,u)-plane.

The following two results are well-known (see, e.g., [11]):

PROPOSITION 2: *The number of connected components of $\tilde{\Phi}^{(i)}_\gamma, i = 1, 2, 3$ is bounded by a constant $N(d) = (d+1)(2d+2)$ depending only on d, the degree of the polynomial f.**

PROPOSITION 3: *For a polynomial $\partial f/\partial u$ of degree $d-1$ the number of the critical values (i.e., the values of $\partial f/\partial u$ at points where grad $\partial f/\partial u = 0$) does not exceed $(d-2)^2$.*

In what follows we can assume, without loss of generality, γ to be a regular value of the polynomial $\partial f/\partial u$, and hence the connected components of $\tilde{\Phi}^{(2)}_\gamma$ to be smooth curves.

The following Definition 3 and Lemmas 1 and 2 form our main tool to study the geometry of γ-critical trajectories.

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* One can get better bounds in Propositions 2 and 3, and Lemma 1, using specific two-dimensional considerations, in particular, Harnack's theorem.

Definition 3: For any point $(\bar{x}, \bar{u}) \in \tilde{\Phi}_\gamma^{(1)}$ define the set $D_\gamma(\bar{x}, \bar{u})$ as follows: $(x, u) \in D_\gamma(\bar{x}, \bar{u})$ if there is a piecewise smooth curve L in the (x, u)-plane connecting the points $(\bar{x}, \bar{u})$ and (x, u) and satisfying the conditions:

(a) $L \subseteq \tilde{\Phi}_\gamma^{(1)}$,

(b) L consists of vertical segments $x = x_i$ and smooth curves $u = \varepsilon_i(x)$, $x_i \leq x \leq x_{i+1}$,

(c) for any $\xi \in [\bar{x}, x], \xi \neq x_i$, the line $x = \xi$ intersects L at one point. If the conditions (a), (b), (c) hold, we call L a curve of type A. ∎

LEMMA 1: *If $(x, u) \in D_\gamma(\overline{x}, \overline{u})$ then there exists a curve L_1 of type A connecting the points (x, u) and $(\overline{x}, \overline{u})$ and such that:*

(a) *the number of vertical segments is at most $N_4(d)$ where $N_4(d) = (2d - 1)(4d - 3) + 2$;*

(b) *the smooth curves $u = \varepsilon_i(x), x_i \leq x \leq x_{i+1}$, are segments of the boundary of the set $\tilde{\Phi}_\gamma^{(1)}$.*

Proof: Let L be a curve of type A connecting the points $(\overline{x}, \overline{u})$. Suppose that $x > \overline{x}$. For any ξ, $\overline{x} \leq \xi \leq x$, consider the biggest vertical segment whose lowest point is on the curve L and which is contained in $\tilde{\Phi}_\gamma^{(1)}$.

The upper points of all such segments form a piecewise continuous curve L, consisting of parts of the boundary of $\tilde{\Phi}_\gamma^{(1)}$ (see Fig. 1). By fitting in vertical segments in the cuts we obtain a piecewise smooth curve L_1. Each vertical segment which connects the cut points for $\overline{x} < x_i < x$ has at least two common points with the boundary of $\tilde{\Phi}_\gamma^{(1)}$. Let $M_1 = (\xi, \eta)$ be the closest (along the line $x = \xi$) such point to the curve L. Then the vertical line $x = \xi$ is tangent to $\tilde{\Phi}_\gamma^{(2)}$ at point M_1. We show that the number of such vertical tangents is bounded in terms of d.

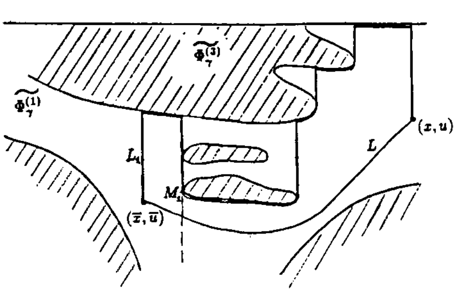

Fig. 1

The coordinates of $M_1 \in D$ satisfy the equations

$$|\frac{\partial f}{\partial u}| = \gamma, \qquad \frac{\partial^2 f}{\partial u^2} = 0.$$

The number of connected components of the set

$$A = \tilde{\Phi}_\gamma^{(2)} \cap \left\{ \frac{\partial^2 f}{\partial u^2} = 0 \right\}$$

is at most $(2d-1)(4d-3)$ [11]. Let C be one of such components and $x = x(t), y = y(t)$ its parametrization. Since γ is a regular value of $\partial f/\partial u$ ($\operatorname{grad} \partial f/\partial u \neq 0$) and $\partial^2 f/\partial u^2 = 0$ we have $\dot{x}(t) = 0$. Hence for any component of A the x coordinate is constant and the number of vertical tangents is at most $(2d-1)(4d-3d)$. Then $N_4(d) = (2d-1)(4d-3)+2$. ∎

LEMMA 2: *Let $(x(\cdot), u(\cdot))$ and $(y(\cdot), v(\cdot))$ be two admissible trajectories such that for any $t \in [0,1]$*

$$(y(t), v(t)) \in D_\gamma(x(t), u(t)).$$

Then

$$|y(t) - x(t)| \leq 2T_2 e^{M_{10}\tau} N_5(d)\gamma$$

where $N_5(d) = 8d^2 - 9d + 4$ and M_{10}, T_2 as in Definition 2.

Proof: Let L be a curve as in Lemma 1 that connects the points $\overline{M} = (x(t), u(t))$ and $M = (y(t), v)t))$. Denote N_i', N_i'' the ends of the vertical segments of L. Then

$$|f(M) - f(\overline{M})| \leq \sum_i{}^{(1)} |f(N_i'') - f(N_i')| + \sum_i{}^{(2)} |f(x_{i+1}, \varepsilon_i(x_{i+1})) - f(x_i, \varepsilon_i(x_i))|.$$

Lagrange's theorem provides the inequality

$$\sum_i{}^{(1)} |f(N_i'') - f(N_i')| \leq \gamma \cdot 2T_2 \cdot N_4(d).$$

On each smooth segment $u = \varepsilon_i(x)$, $x_i \leq x \leq x_{i+1}$, we have:

$$\begin{aligned} |f(x_{i+1}, \varepsilon_i(x_{i+1})) - f(x_i, \varepsilon_i(x_i))| &= \left| \int_{x_i}^{x_{i+1}} \frac{d}{d\xi} f(\xi, \varepsilon_i(\xi) d\xi \right| \\ &= \left| \int_{x_i}^{x_{i+1}} \left[\frac{\partial f}{\partial x}(\xi, \varepsilon_i(\xi)) + \frac{\partial f}{\partial u}(\xi, \varepsilon_i(\xi)) \varepsilon_i'(\xi) \right] d\xi \right| \\ &\leq M_{10} |x_{i+1} - x_i| + \gamma \int_{x_i}^{x_{i+1}} |\varepsilon_i'(\xi)| d\xi. \end{aligned}$$

In the second summand, $\int_{x_i}^{x_{i+1}} |\varepsilon_i'(\xi)|d\xi$ is the variation of $\varepsilon_i(x)$. It is known that for a continuous function ε_i

$$\bigvee_{x_i}^{x_{i+1}} (\varepsilon_i) = \int_{m_i}^{M_i} N_i(\eta)d\eta,$$

where $m_i = \min \varepsilon_i(x), M_i = \max \varepsilon_i(x)$ and $N_i(\eta)$ is the number of roots of the equation $\varepsilon_i(x) = \eta$. Since the curves $\varepsilon_i(x)$ are parts of the boundary of $\tilde{\Phi}_\gamma^{(1)}$, for any $|\eta| < T_2$ we have $\sum_i N_i(\eta) \le d-1$. Then

$$\sum_i \int_{x_i}^{x_{i+1}} |\varepsilon_i'(\xi)|d\xi \le 2T_2(d-1)$$

and

$$\sum\nolimits^{(2)} |f(x_{i+1}, \varepsilon_i(x_{i+1})) - f(x_i, \varepsilon_i(x_i))| \le M_{10}|y(t) - x(t)| + 2T_2(d-1)\gamma.$$

Thus for $\Delta x(t) = y(t) - x(t), t \in [0,1]$, the following differential inequality holds:

$$|\Delta \dot{x}(t)| \le M_{10}|\Delta x(t)| + 2T_2 N_5(d)\gamma, \qquad \Delta x(0) = 0,$$

where

$$N_5(d) = N_4(d) + d - 1 = 8d^2 - 9d + 4.$$

But then we have

$$|\Delta x(t) \le 2T_2 N_5(d) e^{M_{10}t}\gamma. \qquad \blacksquare$$

Definition 4: An admissible trajectory $(x(\cdot), y(\cdot))$ is said to be of the first type if the derivative $\dot{x}(t)$, $t \in [0,1]$, does not change its sign. Otherwise it will be called an admissible trajectory of the second type. ∎

We denote by $\tilde{\Sigma}_1(J,\gamma)$ the set of all γ-critical controls $u(\cdot) \in \tilde{\Sigma}(J,\gamma)$ such that $(x(\cdot), u(\cdot))$ is of the first type. Let $\tilde{\Delta}_1(J,\gamma) = J(\tilde{\Sigma}_1(J,\gamma))$.

Definition 5: Two controls $u(\cdot), v(\cdot)$, defining the admissible trajectories $(x(\cdot), u(\cdot))$ and $(y(\cdot), v(\cdot))$ of the first type, are said to be in the same component of $\tilde{\Sigma}_1(J,\gamma)$ if for any vertical line $x = c$ which intersects both curves $(x(\cdot), u(\cdot))$ and $(y(\cdot), v(\cdot))$ at points P_1 and P_2 respectively, the segment $[P_1, P_2] \subseteq \tilde{\Phi}_\gamma^{(1)}$. ∎

LEMMA 3: *Let $u(\cdot)$ and $v(\cdot)$ belong to the same component of $\tilde{\Sigma}_1(J,\gamma)$. Then for any $t \in [0,1]$, $(y(t),v(t)) \in D_\gamma(x(t),u(t))$.*

Proof: Assume that $y(t) \geq x(t)$. Let P_1 and P_2 be the points of intersection of the two curves with the line $x = x(t)$. The curve L, required in Definition 3, is given by a vertical segment $[P_1, P_2]$ and then by a part of the second curve from P_2 to $(y(t), v(t))$. ∎

From Lemma 2 we obtain the following:

COROLLARY 3: *For any two controls $u(\cdot), v(\cdot)$ which belong to the same component of $\tilde{\Sigma}_1(J,\gamma)$,*

$$|J(u) - J(v)| \leq 2T_2 e^{M_{10}} N_5(d)\gamma.$$

THEOREM 2: *The set $\tilde{\Delta}_1(J,\gamma)$ can be covered by $N_6(d)$ intervals of length $4T_2e^{M_{10}}\gamma_1$, where*

$$\gamma_1 = C_2\gamma^{q/(q+1)} \quad \text{and} \quad N_6(d) = N_5(d)[N_4(d)+1]^2 \cdot 2^{N_3(d)} \leq 8^3 d^6 2^{2(d+1)^2}.$$

Proof: Let $u(\cdot) \in \tilde{\Sigma}_1(J,\gamma)$. Then by Corollary 2 $(x(\cdot),u(\cdot)) \in \tilde{\Phi}_1^{(1)}$. By Corollary 3 it is enough to prove that $\tilde{\Sigma}_1(J,\gamma)$ consists of at most $[N_4(d)+1]^2 \cdot 2^{N_3(d)}$ components according to Definition 5. Denote $\ell = (x(\cdot),u(\cdot))$ and let C be one of the connected components of $\tilde{\Phi}_{\gamma_1}^{(3)}$. Consider all the vertical lines that pass through the points of ℓ. We have the following possibilities:

(a) None of the lines intersects C.
(b) All intersection points lie on one side of ℓ (above or below).
(c) Some line that passes through $(x(\bar{t}),u(\bar{t})) \in \ell$ intersects C on both sides of ℓ.

In the last case let P_1 and P_2 be the intersection points that are closest to ℓ (from above and from below, respectively) and let $\Gamma \in \tilde{\Phi}_{\gamma_1}^{(2)}$ be a continuous curve that connects them. The segment $[P_1,P_2]$ and Γ form a closed curve that bounds some subset G of the (x,u) plane, which we will call a bay for the curve ℓ. The curve ℓ which corresponds to the process of the first type can intersect $[P_1,P_2]$ only once (see Fig. 2). Hence ℓ lies in the subset G either for all $t \geq \bar{t}$, or for all $t \leq \bar{t}$. This means that for any curve ℓ there might be at most two components of $\tilde{\Phi}_{\gamma_1}^{(3)}$ where the possibility (c) takes place.

Further, if a smooth curve $\Gamma \subseteq \Phi_{\gamma_1}^{(2)}$ connects the ends of a vertical segment $[P_1,P_2]$, then it has a point distinct from P_1, P_2 where $\partial^2 f/\partial u^2 = 0$. As we proved above, the number of such points does not exceed $N_4(d) = (2d-1)(4d-3)+2$. Denote these points by $z_1, \ldots, z_p, p \le N_4(d)$.

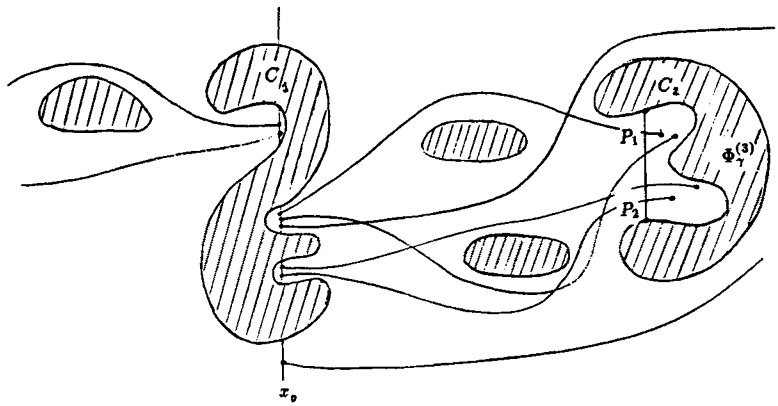

Fig. 2

Now to each admissible trajectory ℓ in $\tilde{\Phi}^{(1)}_{\gamma_1}$, we associate a symbol $\sigma(\ell)$ as follows: Let $C_1, \ldots, C_n$ be the connected components of $\tilde{\Phi}^{(3)}_{\gamma_1}$. We define $\sigma_i(\ell)$, $i = 1, \ldots, n$, to be 0, if the possibilities (a) or (c) are satisfied for C_i and ℓ. $\sigma_i(\ell)$ is defined to be 0 also in the case (b), if ℓ passes under C_i. In the case (b), if ℓ passes above $C_i, \sigma_i(\ell)$ is defined to be 1.

If ℓ starts (ends) in a bay, we define $\sigma_{n+1}(\ell)$ ($\sigma_{n+2}(\ell)$) to be the smallest number j such that the "vertical tangent" point z_j lies on the boundary of this bay. If ℓ does not start (end) in a bay, $\sigma_{n+1}(\ell)$ ($\sigma_{n+2}(\ell)$) is 0.

LEMMA 4: *If $\sigma(\ell_1) = \sigma(\ell_2)$, then ℓ_1 and ℓ_2 are in the same connected component of $\tilde{\Sigma}_1(J, \gamma_1)$.*

Proof: Assume that there is a vertical line $x = c$, intersecting ℓ_1 and ℓ_2 at the points P_1 and P_2, respectively, such that the segment $[P_1, P_2]$ intersects $\tilde{\Phi}^{(3)}_{\gamma_1}$. Then either ℓ_1 and ℓ_2 pass on different sides of the same component C_q, and then $\sigma_q(\ell_1) \ne \sigma_q(\ell_2)$, or they start (end) in different bays, and then they will differ in one of their two last coordinates.

Since, clearly, the number of different symbols $\sigma(\ell)$ does not exceed

$$2^n \cdot (p+1)^2 \le 2^{N_3(d)} \cdot [N_4(d)+1]^2,$$

where

$$N_3(d) = (d+1)(2d+2), \qquad N_4(d) = (2d-1)(4d-3)+2,$$

the same upper bound holds for the number of components of $\tilde{\Sigma}_1(J, \gamma_1)$.

By Corollary 3, the image of each component can be covered by an interval of length $4T_2e^{M_{10}}N_5(d)\gamma_1$, hence by $N_5(d)$ intervals of length $4T_2e^{M_{10}}\gamma_1$. Thus all the set $\tilde{\Delta}_1(J, \gamma)$ can be covered by

$$N_6(d) = N_5(d)[N_4(d) + 1]^2 2^{N_3(d)} \leq 8^3 d 2^{2(d+1)^2}$$

intervals of length

$$4T_2e^{M_{10}}\gamma_1 = 4T_2e^{M_{10}}[(q+1)K_1e^{q(m_{10}+M_{10})}]^{1/(q+1)} \cdot \gamma^{q/(q+1)}.$$

Theorem 1 is proved. ∎

Now we study the structure of the trajectories of the second type.

Lemma 5: *The boundary of the set $D_\gamma(x, u)$ contains only two types of segments:*
(a) *segments of the boundary of $\tilde{\Phi}_\gamma^{(1)}$,*
(b) *vertical segments tangent to $\tilde{\Phi}_\gamma^{(2)}$.*

Proof: Suppose the contrary: let (ξ, η), a boundary point of the set $D_\gamma(x, u)$, be an internal point for the set $\tilde{\Phi}_\gamma^{(1)}$ and the line $x = \xi$ is not tangent to the set $\tilde{\Phi}_\gamma^{(2)}$ (see Fig. 3). Let P_1, P_2 be the intersection points of the line $x = \xi$ with the boundary of the set $\tilde{\Phi}_\gamma^{(1)}$ that are closest to the point (ξ, η) (one from above and one from below). Then there exists an $\varepsilon > 0$ such that:
(a) the segments of the boundary that pass through the points P_1, P_2 can be given by $u = u_1(x)$, $u = u_2(x)$, $x \in [\xi - \varepsilon, \xi + \varepsilon]$;
(b) the vertical lines $x = \xi \pm \varepsilon$ and the curves $u = u_1(x)$, $u = u_2(x)$ bound a curvilinear rectangle P_1'', P_2'', P_2', P_1', so that all the internal points of the rectangle are also the internal points of $\tilde{\Phi}_\gamma^{(1)}$.

Suppose that $x < \xi$. In order to reach the point (ξ, η), a curve of type A that connects (x, u) with (ξ, η) must intersect the segment $[P_1'', P_2'']$ at some point Q. Continue the type A curve after the point Q in the following way: first along the vertical line $x = \xi - \varepsilon$, then along the horizontal lines $u = \eta \pm \delta$. In this fashion we can cover with type A curves a rectangle centered at (ξ, η) — and this contradicts the premise that (ξ, η) is a boundary point of the set $D(x, u)$.

Theorem 3: *If $(x(\cdot), u(\cdot))$ is an admissible trajectory of the second type, then*

$$|x(1) - x(0)| \leq 4T_2e^{M_{10}}N_7(d)\gamma_1$$

where

$$\gamma_1 = C_2\gamma^{q/(q+1)} \quad \text{and} \quad N_7(d) = N_5(d)[N_4(d)+1].$$

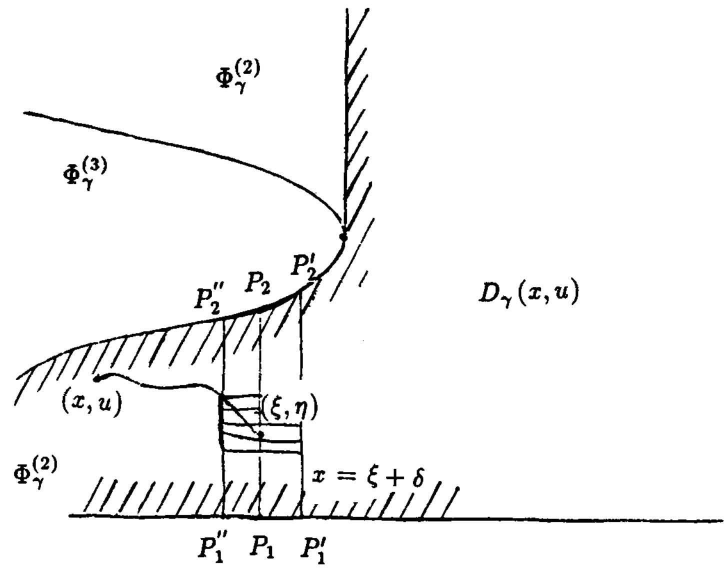

Fig. 3

Proof: If $(x(\cdot),u(\cdot))$ is a trajectory of the second type, then for some t^*, $0 \le t^* \le 1$ there holds $\dot{x}(t^*) = 0$. Then equation (2) has a stationary solution $x = x(t^*)$, $u = u(t^*)$. Let t_1^* be the first among such points. Applying Lemma 3 to the trajectories $(x(t),u(t))$ and $(x \equiv x(t_1^*),\ u \equiv u(t_1^*))$, we have $(x(t),u(t)) \in D_{\gamma_1}(x(t_1^*),u(t_1^*))$ for $t \in [0,t_1^*]$. By Lemma 2 we now have

$$|x(0) - x(t_1^*)| \le 2T_2 e^{M_{10}} N_5(d)\gamma_1.$$

Denote $\bar{t}_1 = \sup\{\tau | (x(t),u(t)) \in D_{\gamma_1}(x(t_1^*),u(t_1^*)) \text{ for } t \in [t_1^*,\tau]\}$. Then the following holds:

(a) $|x(\bar{t}_1) - x(0)| \le |x(\bar{t}_1) - x(t_1^*)| + |x(t_1^*) - x(0)| \le 4T_2 e^{M_{10}} N_5(d)\gamma_1$;

(b) the point $(x(\bar{t}_1),u(\bar{t}_1))$ lies in one of the vertical tangents to the set $\tilde{\Phi}_{\gamma_1}^{(2)}$ (the proof follows from Lemma 5);

(c) There exists a $t', t_1^* < t' \le \bar{t}_1$, such that $\dot{x}(t') = 0$.

Indeed, if this was false, then for some sufficiently small ε and $t \in [t_1^*, \bar{t}_1 + \varepsilon]$ the trajectory $(x(\cdot),u(\cdot))$ would be a type A curve and $(x(t),u(t)) \in D_{\gamma_1}(x(t_1^*),u(t_1^*))$ for $t \in [t_1^*, \bar{t}_1 + \varepsilon]$ which contradicts the choice of $\bar{t}_1$.

Denote $t_2^* = \sup\{t'|\dot{x}(t') = 0, t' \in [t_1^*, \bar{t}_1]\}$. For t_2^* there exists $\bar{t}_2$ such that the point $(x(\bar{t}_2), u(\bar{t}_2))$ lies in a vertical tangent to the set $\tilde{\Phi}_{\gamma_1}^{(2)}$ and $(x(t), u(t)) \in D_{\gamma_1}(x(t_2^*), u(t_2^*))$ for $t \in [t_2^*, \bar{t}_2]$. Then by Lemma 2 we have

$$|x(\bar{t}_2) - x(\bar{t}_1)| \leq |x(\bar{t}_2) - x(t_2^*)| + |x(t_2^*) - x(\bar{t}_1)| \leq 4T_2 e^{M_{10}} N_5(d)\gamma_1.$$

Continue the above process. At each step the trajectory $(x(\cdot), u(\cdot))$ can pass from one vertical tangent $x = x(\bar{t}_n)$ to another $x = x(\bar{t}_{n+1})$, while

$$|x(\bar{t}_n) - x(\bar{t}_{n+1})| \leq 4T_2 e^{M_{10}} N_5(d)\gamma_1.$$

Now let Ω_n be the set of abscissas of vertical tangents crossed during this process till time t_n (some of them can be crossed more than once). The last inequality shows that

$$\operatorname{diam} \Omega_{n+1} \leq \operatorname{diam} \Omega_n + 4T_2 e^{M_{10}} N_5(d)\gamma_1.$$

Since the cardinality of Ω_n can never exceed $N_4(d)$, we have

$$|x(\bar{t}_p) - x(0)| \leq 4T_2 e^{M_{10}} N_5(d) N_4(d)\gamma_1$$

where $\bar{t}_p$ is the last point of the $\bar{t}_n$. (In particular, this means that our trajectory can visit only those vertical tangents which are "near one another".)

Therefore $|x(1) - x(0)| \leq 4T_2 e^{M_{10}} N_7(d)\gamma_1$ where $N_7(d) = N_5(d)[N_4(d) + 1]$. Theorem 3 is proved. ∎

From the Theorems 2 and 3, Theorem 1 of Section 2 follows immediately:

THEOREM 1: *The set $\tilde{\Delta}(J, \gamma)$ can be covered by $N(d)$ intervals of length $4T_2 e^{M_{10}}\gamma_1$, where*

$$N(d) = N_6(d) + 2N_7(d) \leq (8d^2 - 9d + 7)^3 d^6 2^{2(d+1)^2} \leq 8^3 d^6 2^{2(d+1)^2}.$$

4. Examples

Next we give examples which show that the bound given in Theorem 1 is essentially sharp.

Example 1: Let $T_d(x)$ be a Chebyshev polynomial of degree d:

$$T_d(x) = \cos(d \arccos x).$$

The polynomial $T_d(x)$ has zeros in $x_k = \cos(2k-1)\pi/2d$, $k = 1, \ldots, d$ and extrema in $x_k = \cos k\pi/d$, $k = 0, 1, \ldots, d$.

Set $\tilde{T}_d(x) = T_d(2x-1)$ and

$$\frac{\partial f}{\partial u} = (u - \tilde{T}_d(x))(u - \tilde{T}_d(x) + \xi T_d(u)).$$

The curve $L_2 = u - \tilde{T}_d(x) - \xi T_d(u) = 0$ lies close to the curve $L_1 = u - \tilde{T}_d(x) = 0$ for small values of ξ and crosses it in d^2 points for $0 < x < 1$. The set

$$\phi_\gamma^{(1)} = \left\{ (x,u) \middle| \left| \frac{\partial f}{\partial u} \right| \leq \gamma \right\}$$

is shown in Fig. 4.

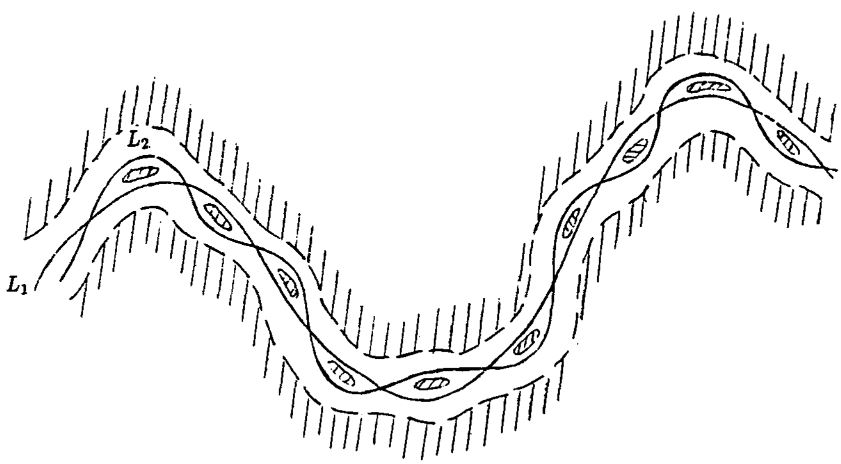

Fig. 4

Define

$$f_1(x,u) = c\left[\int \frac{\partial f}{\partial u} du + \varepsilon(x) \right]$$

where $c > 0$ and the function $\varepsilon(x) > 0$ will be defined later. We can assume $f_1(x,y)$ to be positive in the rectangle $D = \{|x| \leq 2, |u| \leq 2\}$.

For $0 < x < 1$ the curves L_1 and L_2 are divided by the crossing points into $d^2 - 1$ arcs each, denoted respectively as $\ell_1^{(i)}$ and $\ell_2^{(i)}$. Consider all the possible paths obtained by choosing either $\ell_1^{(i)}$ or $\ell_2^{(i)}$ for each i. There are 2^{d^2-1} such paths. Each of them can become a critical trajectory for a suitable choice of a control.

Indeed, by substituting the explicit equation of such a curve $u = \psi(x)$ into (2), we obtain a solution $x = x_\psi(t)$ of the equation

$$\dot{x} = f(x, \psi(x))$$

that corresponds to a control $u(t) = \psi(x_\psi(t))$. The pair $(x(\cdot), \psi(x_\psi(\cdot)))$ is critical.

The time it takes the point $(x(t), u(t))$ to travel along the arc $\ell_1^{(i)}$ that connects the points (x_i, u_i) and (x_{i+1}, u_{i+1}) is

$$\int_{x_i}^{x_{i+1}} \frac{dx}{f_1(x, \psi_1^{(i)}(x))}.$$

Since $\partial f_1/\partial u < 0$ between the arcs $\ell_1^{(i)}$ and $\ell_2^{(i)}$, the time of passage over the upper arcs is greater than for the lower arcs.

One can make the difference in passage time from (x_i, u_i) to $(x_{i+1}, u_{i+1}), i = 1, \dots, d^2 - 1$ equal to $1/2^{i+1}$ by an appropriate choice of the function $\varepsilon(x)$. For example, set $h(x) = e^{-1/(1-x^2)}$ for $-1 < x < 1$ and $h(x) = 0$ for all other values of x and let

$$\varepsilon(x) = \sum_{i=1}^{d^2-1} \xi_i h\left(\frac{2}{x_{i+1} - x_i}\left(x - \frac{x_i + x_{i+1}}{2}\right)\right).$$

Choose the numbers ξ_i and c so that

$$\left|\int_{x_i}^{x_{i+1}} \left[\frac{1}{cf_1(x, \psi_1^{(i)}(x))} - \frac{1}{cf_1(x, \psi_2^{(i)}(x))}\right] dx\right| = \frac{1}{2^{i+1}}, \quad i = 1, \cdots, d^2 - 1.$$

Such a choice of $\varepsilon(x)$ will not affect the polynomial character of $\partial f/\partial u$. (We use ε in this form to simplify computations. More detailed analysis allows one to construct the same example with f being a polynomial.)

Now set $f = c_1 f_1$ and choose a number $c_1 > 1$ so that the time of passage over the lower arcs from $(0,1)$ to $(1,1)$ will be

$$\tau = \int_0^1 \frac{dx}{c_1 f_1} \le \frac{1}{2}.$$

For different trajectories ψ that are glued together from the arcs $l_1^{(i)}$ and $l_2^{(i)}$, the point (1,1) will be reached in different times τ_ψ,

$$\tau \leq \tau_\psi < \tau + \frac{1}{2c_1} < 1.$$

Moreover, by construction, these time moments τ_ψ divide the interval $[\tau, \tau + 1/2c_1]$ into 2^{d^2-1} equal parts.

After the point (1,1) we send all the critical trajectories along the curve L_1. Since all of the 2^{d^2-1} trajectories pass the point (1,1) at different times, they will stop at the time $t = 1$ in different final points whose x coordinates will all be distinct and also uniformly distributed, up to a bounded distortion.

As a result we obtain 2^{d^2-1} different and "almost uniformly" distributed critical values of the functional $J(u)$. The allowing for γ-critical controls gives us as $\tilde{\Delta}(J,\gamma)$ 2^{d^2-1} segments around these critical values.

Let $M(\gamma, A)$ be a minimal number of intervals of length 2γ that cover a set $A \subseteq \mathbf{R}$. Then for $\gamma > 0$ sufficiently small, namely, $\gamma \leq 1/2^{d^2}$, we have

$$M(\gamma_1, \tilde{\Delta}(J,\gamma)) \sim 2^{d^2}.$$

The following example shows how the trajectories of type (c), as described in Theorem 1, can appear.

Example 2: Let (x^*, u^*), $x^* > 1$, be a point that each of the 2^{d^2-1} critical trajectories eventually reaches. We multiply the polynomial $\partial f/\partial u$ described above by a polynomial

$$\tilde{g}(x,u) = g(x - x^*, u - u^*)$$

where

$$g(x,u) = \prod_{i=1}^{d}(x - \alpha_i u) + M(x^{d+2} + u^{d+2})$$

and $M > 0$ is sufficiently big (d is even).

We denote by L_3 the set of zeros of the polynomial $\tilde{g}(x,u)$. The set $\Phi_\gamma^{(1)}$, for the function

$$\frac{\partial f_2}{\partial u} = \frac{\partial f}{\partial u} \cdot \tilde{g},$$

is shown in Fig. 5.

Let the critical trajectories that reach the point (x^*, u^*) continue along different arcs of the curve L_3. The parameters $\alpha_1, \dots, \alpha_d$ can be chosen in such a way that all the values of $x(1)$ will be different. Then the number of distinct critical values will be $2d \cdot 2^{d^2-1}$.

By passing to γ-critical controls we obtain for a sufficiently small $\gamma > 0$

$$M(\gamma_1, \tilde{\Delta}(J,\gamma)) \sim d \cdot 2^{d^2}.$$

Using a similar construction for the beginning of the trajectories one can increase $M(\gamma_1, \tilde{\Delta}(J,\gamma)$ by a further factor of d.

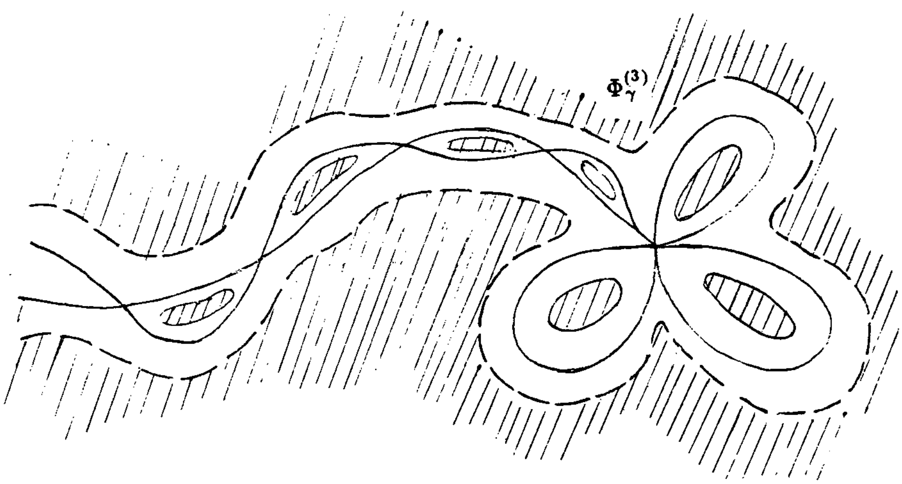

Fig. 5

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